Crumpling transition and flat phase of polymerized phantom membranes

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Polymerized phantom membranes are revisited using a nonperturbative renormalization group approach. This allows one to investigate both the crumpling transition and the low-temperature, flat, phase in any internal dimension D and embedding dimension d, and to determine the lower critical dimension. The crumpling phase transition for physical membranes is found to be of second order within our approximation. A weak first-order behavior, as observed in recent Monte Carlo simulations, is however not excluded.

Membranes form a particularly rich and exciting domain of statistical physics in which the interplay between two-dimensional geometry and thermal fluctuations has led to a lot of unexpected behaviors going from flat to tubular and glassy phases (see [1, 2, 3, 4] for reviews). Roughly speaking, membranes fall into two groups [4]: fluid membranes, in which the building monomers are free to diffuse. The connectivity is thus not fixed and the membrane displays a vanishing shear modulus. In contrast, in polymerized membranes the monomers are tied together through a potential which leads to a fixed connectivity and to elastic forces. While fluid membranes are always crumpled, polymerized membranes, due to their nontrivial elastic properties, exhibit a phase transition between a crumpled phase at high temperature and a flat phase at low temperature with orientational order between the normals of the membrane [4, 5, 6, 7]. Amazingly, due to the existence of long-range forces mediated by phonons, the correlation functions in the flat phase display a nontrivial infrared scaling behavior [8, 9, 10]. Accordingly, the lower critical dimension above which an order can develop appears to be smaller than 2 [10], in apparent violation of the Mermin-Wagner theorem.

Let us consider the general case of D-dimensional non self-avoiding (phantom) membranes embedded in a d-dimensional space. Early ϵ-expansion [7] performed at one-loop order on the Landau-Ginzburg-Wilson-type model relevant to study the crumpling transition of polymerized membranes has led to predict that just below the upper critical dimension $D = 4$, the crumpling transition is of second order for $d > d_{cr} = 219$ while it is of first order for $d < d_{cr}$. This leaves however open the question of the nature of the transition in the physical $(D = 2, d = 3)$ situation, the case $\epsilon = 2$ being clearly out of reach of such a one-loop order computation. On the numerical side former Monte Carlo (MC) studies (see [11, 12] for reviews) predict a second-order behavior while more recent simulations [13, 14] rather favor first-order behaviors. There is however no definite conclusion and no explanation for these versatile results.

In parallel to the investigation of the crumpling tran-

sition, an effective elastic field theory has been used to probe the flat, low-temperature, phase of membranes [4, 5, 8, 10]. An ϵ-expansion has been performed [8], also at one-loop order, below the upper critical dimension $D = 4$ showing that this flat phase is controlled by a nontrivial fixed point (FP). However, again, this low order computation performed in the vicinity of $D = 4$ has been of no use to accurately determine the properties of genuine $2D$ membranes such as the critical exponents and the lower critical dimension $D_{lc}(d)$ above which the flat phase can exist.

Significant progress has been realized with the use of large-d expansion [10, 15], and variant of, such as self-consistent screening approximations (SCSA) [16] that have allowed to evaluate the exponent η both at the crumpling transition and in the flat phase as well as the dimension $D_{lc}(d)$. However, the very nature of the approach, requiring large values of d, makes doubtful the quantitative predictions extrapolated at small d and even impossible the determination of the line $d_{cr}(D)$, separating the first-order from the second-order regions.

A flaw of the previous approaches to polymerized membranes is that, due to their perturbative character, they are unable to treat all aspects of the physics of membranes including crumpling transition, flat phase, and lower critical dimension and thus to get a global picture of the renormalization group (RG) phase diagram. In this article, we propose an approach of polymerized membranes based on a nonperturbative RG method [17] which has been applied successfully in both particle and condensed matter physics (see [18, 19, 20] for reviews). With this method, that we adapt to the treatment of extended objects, we are able to describe within the *same* formalism, *i.e.*, using a *unique* effective action and a *unique* set of RG equations, both the crumpling transition and the flat phase of membranes. Concerning the crumpling transition, we reproduce the results obtained within the ϵ-expansion approach and at leading order within the large-d approaches. Moreover, we determine the line $d_{cr}(D)$ everywhere between $D = 4$ and $D = 2$. Our estimates of $d_{cr}(D = 2) \simeq 2$ lead to predict a second-order phase tran-

sition in the physical case but do not completely exclude a weak first-order behavior. Our investigation of the flat phase also allows to recover all previous perturbative results including ϵ and $1/d$-expansions. Moreover we get, *for all values of* d, a determination of the lower critical dimension $D_{lc}(d)$ above which the crumpling transition and flat phase fixed points are shown to coexist.

Our approach is based on the concept of effective average action [17] (see [18, 19, 20] for reviews), $\Gamma_k[\mathbf{r}]$, where $\mathbf{r} = \mathbf{r}(\mathbf{x})$ is a d-dimensional *external* vector that describes the membrane in the embedding space while $\mathbf{x}$ is a set of *internal* D-dimensional coordinates which labels a point within the membrane. The quantity $\Gamma_k[\mathbf{r}]$, k being a running scale going from a lattice scale $k = \Lambda$ to the infrared scale $k = 0$, has the physical meaning of a coarse grained free energy where only fluctuations with momenta $q \geq k$ have been integrated out. Thus, at the lattice scale Λ, $\Gamma_{k=\Lambda}$ identifies with the continuum limit of some lattice Hamiltonian while at long distance, *i.e.*, at $k = 0$, it identifies with the standard free energy Γ. The k-dependence, RG flow, of Γ_k is provided by an exact evolution equation [17],

$$\frac{\partial \Gamma_k}{\partial t} = \frac{1}{2} \text{Tr} \left\{ (\Gamma_k^{(2)} + R_k)^{-1} \frac{\partial R_k}{\partial t} \right\} \qquad (1)$$

where $t = \ln k/\Lambda$. The trace has to be understood as a D-dimensional momentum integral as well as a summation over internal indices. In Eq.(1), $R_k(q)$ is an effective infrared cut-off function which suppresses the propagation of modes with momenta $q < k$ and makes that Γ_k encodes only modes with momenta $q \geq k$. A convenient cut-off is provided by $R_k(q) = Z(k^4 - q^4)\theta(k^2 - q^2)$, where Z is a field renormalization – see below. It generalizes to translational-invariant action density a cut-off [21] which has been largely used since it leads to compact expressions. Finally, and of utmost importance, note that the term $\Gamma_k^{(2)}$ in Eq.(1) is, in principle, the exact, *i.e.*, *full field-dependent*, inverse propagator, the second derivative of Γ_k with respect to the field $\mathbf{r}$, taken in a *generic*, nonvanishing field configuration. This is the fact at the very origin of the nonperturbative character of the method.

Let us now make precise the form of Γ_k. It must be invariant under the group of Euclidean displacements which includes translations and rotations. This imposes to Γ_k to be a functional of $\partial_\alpha \mathbf{r} \equiv \partial \mathbf{r}/\partial x_\alpha$, $\alpha = 1 \ldots D$, the order parameter, and of scalars in both the embedding and membrane spaces. An exact treatment of Eq.(1) would imply Γ_k to enclose all powers and derivatives of these Euclidean invariants. This goal is however unrealistic and one has to truncate Γ_k. We choose here an ansatz that allows both to make easily contact with previous – perturbative – approaches and to realize our program. It is given by:

$$\Gamma_k(\mathbf{r}) = \int d^D x \quad \frac{Z}{2} \left(\partial_\alpha \partial_\alpha \mathbf{r} \right)^2 + u \left(\partial_\alpha \mathbf{r}.\partial_\beta \mathbf{r} - \zeta^2 \delta_{\alpha\beta} \right)^2$$
$$+ \quad v \left(\partial_\alpha \mathbf{r}.\partial_\alpha \mathbf{r} - D\zeta^2 \right)^2 \qquad (2)$$

where Z, u, v and ζ are the running couplings which parametrize the model, with the indices α and β running over $1 \ldots D$. This is, up to a redefinition of the couplings, the action used in [7] to investigate the crumpling transition. Let us recall the physics encoded in Eq.(2) at the mean-field level with $u > 0$ and $u + vD > 0$. For $\zeta^2 = 0$, the minimum of Γ_k is given by a configuration where $\partial_\alpha \mathbf{r}$ vanishes which characterizes a crumpled phase. For $\zeta^2 > 0$ this minimum is given by a configuration $\mathbf{r}(\mathbf{x}) = \zeta \sum_{\alpha=1}^D x_\alpha \, \mathbf{e}_\alpha$, where the $\{\mathbf{e}_\alpha\}$ are D orthonormal vectors, which corresponds to a D-dimensional flat phase. Action (2) thus describes a transition between a high-temperature, crumpling, phase and a low-temperature, flat, phase. The excitation spectrum in the ordered phase is provided by $d - D$ out-of-plane, capillary, waves and D in-plane, phonon, modes. A crucial aspect of our approach is that, since we establish nonperturbative RG equations for the couplings entering in Eq.(2), and in particular for the coupling ζ, we are able to tackle both the crumpling transition, typically associated to a vanishing ζ, and the flat phase fixed point (FLFP) which is reached by letting ζ run to infinity.

Technically, the flow equations for the couplings Z, u, v and ζ are obtained using their definitions in terms of functional derivatives of the effective action (see [19, 20, 22] for details) and applying RG Eq.(1). In terms of dimensionless quantities, these equations write as follows:

$$\partial_t \zeta^2 = -(D - 2 + \eta_t) \zeta^2 + \frac{4 A_D}{D} \left\{ (D - 1) \frac{(2u + vD)}{u + vD} l_{010}^{D+2} \right.$$
$$+ \left. \frac{3u + (D+2)v}{u + vD} l_{001}^{D+2} + (d - D) l_{100}^{D+2} \right\}$$

$$\partial_t u = (D - 4 + 2\eta_t) u + \frac{16 A_D}{D(D+2)} \left\{ 2 (3u + 2v)^2 l_{002}^{D+4} \right.$$
$$+ 4D u(u + v) l_{011}^{D+4} + u^2(D^2 + 2D - 8) l_{020}^{D+4}$$
$$+ \left. 2u^2(d - D) l_{200}^{D+4} \right\} \qquad (3)$$

$$\partial_t v = (D - 4 + 2\eta_t) v + \frac{16 A_D}{D(D+2)} \left\{ -4u(u + v) l_{011}^{D+4} \right.$$
$$+ (d - D)\left(u^2 + 2(D+2)uv + D(D+2)v^2 \right) l_{200}^{D+4}$$
$$+ \left((3D+2)u^2 + (D^2 + D - 2)(4uv + Dv^2) \right) l_{020}^{D+4}$$
$$+ \left. \left(9u^2 + 6(D+4)uv + (D^2 + 6D + 12)v^2 \right) l_{002}^{D+4} \right\}$$

where $A_D = 2^{-D-1} \pi^{-D/2} / \Gamma(D/2)$. The flow of Z, that provides the function $\eta_t = -d \ln Z/dt$ giving the critical exponent η at a FP, is too long to be displayed here (see [22]). In Eqs.(3) l_{abc}^D is a shortcut for :

$$l_{abc}^D = \frac{-1}{4 A_D} \widehat{\frac{\partial}{\partial t}} \int d^D q \, \frac{1}{[P_0(q)]^a} \frac{1}{[P_1(q)]^b} \frac{1}{[P_2(q)]^c} \qquad (4)$$

where $P_i(q) = Z q^4 + R_k(q) + m_i^2 q^2, i = 0, 1, 2$ and $\widehat{\partial}/\partial t$ only acts on R_k. These so-called "threshold functions" (see [19, 20]) control the relative role of the different modes, phonons and capillary waves, within the RG flow. In Eq.(4), the mass $m_0 = 0$ is associated to the $d - D$ transversal, capillary, modes while $m_1^2 \equiv 4\zeta^2 u$ and $m_2^2 \equiv 8\zeta^2(u+v)$ are masses associated to the D phonons modes that split up into $D-1$ modes with mass m_1 and one mode with mass m_2.

– *The crumpling transition* – Let us first consider the crumpling transition. To recover the RG equations derived perturbatively in [7] one expands Eq.(3) in powers of both $\epsilon = 4 - D$ and the couplings u and v that are of order ϵ at any putative nontrivial FP. This also corresponds to an expansion in powers of the phonon masses, which are small at the crumpling transition FP. Using the fact that the threshold functions entering in the flow of u and v have a universal, cut-off independent, limit at vanishing masses in $D = 4$ given by $l_{abc}^8 = 1$, one obtains:

$$\partial_t u = -\epsilon u + \frac{(d+21)u^2 + 20vu + 4v^2}{24\pi^2} \tag{5}$$

$$\partial_t v = -\epsilon v + \frac{(d+15)u^2 + 4(3d+17)vu + 4(6d+7)v^2}{48\pi^2}.$$

Up to a change in variable $(v \to v - u/4)$ these are the equations derived in [7]. We recall that, at sufficiently high values of d, *i.e.*, $d > d_{cr} = 219$, just below $D = 4$, the sets of Eqs.(3) and (5) admit a stable (in the u and v directions) FP associated to the crumpling transition, called crumpling transition fixed point (CTFP). Still at $d > d_{cr}$, there exists another FP, close to the CTFP, which is unstable and that, when the dimension d is lowered to d_{cr}, annihilates with the CFTP, defining the curve $d_{cr}(D)$. A large-d analysis of Eqs.(3) can be also easily done. The leading contributions come from the capillary modes which enter in Eq.(3) through the terms proportional to $d - D$. With our cut-off function $l_{100}^D = 4/D$ and $l_{200}^D = 8/D$ so that the coordinates of the CTFP are given by $\zeta_{cr}^2 = 16A_D/D(D^2 - 4)$, $u_{cr} = (16 - D^2)D(2 + D)/(256dA_D)$ and $v_{cr} = -(16 - D^2)D/(256dA_D)$. The corresponding critical exponents are: $\nu = 1/(D - 2) + O(1/d)$ and $\eta = O(1/d)$ in agreement with [6] and [10].

To tackle with the physics below $D = 4$ we have numerically solved the FP equations between $D = 4$ and $D = 2$, a dimension in which the effects of truncation start to be important. The right part of Fig.1 summarizes our results: one finds a smooth curve $d_{cr}(D)$ which starts at $d_{cr} = 219$ in $D = 4$ and reaches $d_{cr} \simeq 2$ in $D = 2$ leading to predict a second-order phase transition for physical membranes. In this last case one finds, at the CTFP, a thermal exponent $\nu = 0.52$ and $\eta = 0.627$ which compares well with the results provided by the large-d expansion $\eta = 2/3$ [6, 10] and MC results $\eta = 0.71(5)$ [23] but less with the Monte Carlo Renormalization Group $\eta = 0.85(15)$ [24] and the SCSA $\eta = 0.535$ [16]. At our

level of approximation, our results display a weak dependence with respect to the cut-off function $R_k(q)$ that induces an error on the curve $d_{cr}(D)$. Using another cut-off, $R_k(q) = Zq^4/(\exp(q^4/k^4) - 1)$, we have evaluated the error bar on $d_{cr}(D = 2)$, which is typically of order $\delta d_{cr} \sim 1$. This means that one cannot exclude $d_{cr}(D = 2)$ to be close to, or even slightly above, $d = 3$ so that the crumpling transition for genuine membranes would be predicted to be of weak first-order in agreement with recent MC results [13, 14]. This point will be further analyzed in the near future [22].
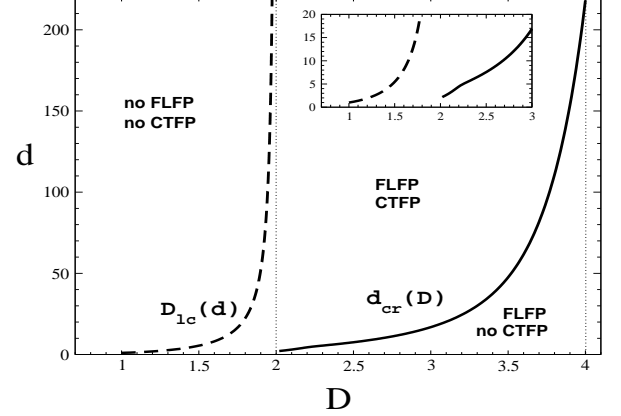

FIG. 1: On the right part, the curve $d_{cr}(D)$ which separates the region with a CTFP and without a CTFP. On the left part, the lower critical dimension $D_{lc}(d)$.

– *The flat phase* – The equations relevant to study the flat phase are easily obtained in our formalism by considering the regime $\zeta \gg 1$ in the RG flow Eqs.(3), which corresponds to a regime where the phonon masses are very large and thus to a regime dominated by the fluctuations of the capillary waves, as expected in the deep flat phase. Setting $\tilde{d} = d - D$ one gets:

$$\partial_t u = (D - 4 + 2\eta_t) u + \frac{256 \, \tilde{d} \, u^2 \, \tilde{A}_D}{D(D+2)(D+4)(D+8)}$$

$$\partial_t v = (D - 4 + 2\eta_t) v +$$

$$\frac{128 \, \tilde{d} \, (u^2 + 2(D+2)uv + D(D+2)v^2)\tilde{A}_D}{D(D+2)(D+4)(D+8)} \tag{6}$$

$$\eta_t = \frac{128(D+4)(D^2-1)u(u+2v)A_D}{(D^4 + 6D^3 + 8D^2)(u+v) + 128(D^2-1)u(u+2v)A_D}$$

with $\tilde{A}_D = A_D(8 + D - \eta_t)$ and, for $\alpha = 1/\zeta^2$:

$$\partial_t \alpha = (D - 2 + \eta_t)\alpha - \frac{16 \, \tilde{d}(6 + D - \eta_t)\alpha^2 A_D}{D(D^2 + 8D + 12)} \tag{7}$$

an equation which generalizes, to any value of D and d, the one obtained in the limit of large elastic constants, $D = 2$ and large-d, in [6]. Note that the function η_t in Eqs.(6) and (7) determines, at a FP, the exponent η of the

capillary waves. The analog exponent, η_u, for the phonon modes, is obtained by the usual Ward identity [10]: $\eta_u = 4 - D - 2\eta$ that follows from rotational invariance.

The set of Eq.(6), when expanded in powers of $\epsilon = 4 - D$, degenerates into those derived perturbatively in [8]. Accordingly, Eqs.(6-7), admit three nontrivial FPs, among which one, the FLFP, is stable with respect to *all* directions including α down to a dimension D_{lc}, the lower critical dimension. In the limit of large codimension $\tilde{d}$ the coordinates of the FLFP are $\alpha_f = 0$ ($\zeta_f^2 \to \infty$), $u_f = (16 - D^2)D(2 + D)/(256dA_D)$ and $v_f = -(16 - D^2)D/(256dA_D)$, with these two last quantities being identical to those of the CTFP. At the FLFP one finds $\eta = O(1/d)$, in agreement with previous large-d approach [10]. Moreover Eq.(7) indicates that, at large d, the FLFP is stable down to $D_{lc}(d \to \infty) = 2$, in agreement with [10] which predicts: $D_{lc}(d \to \infty) = 2 - 2/d + O(1/d^2)$. Note also that the RG flow on u and v indicates that for $\tilde{d} = 0$ and at any nontrivial FP one has the exact result $\eta = (D - 4)/2$ [16]. For physical membranes one finds $\eta = 0.849$ which compares well with the SCSA $\eta = 0.821$ [16] and numerical simulations $\eta = 0.750(5)$ [23] and $\eta = 0.81(3)$ [25] but less with the large-d result $\eta = 2/3$ [10].

Finally Eqs.(6-7) also allow a determination of $D_{lc}(d)$ for *all* values of d. To do this we use the equality [10]: $\eta(D_{lc}, u_f, v_f) = 2 - D_{lc}$ at the FLFP which defines D_{lc} as the dimension at which phonons and capillary waves scale identically [10]. In fact, the RG Eq.(7) provides another interpretation of D_{lc}. Indeed, above D_{lc}, Eq.(7) possesses a solution with $\alpha \neq 0$ which corresponds to the CTFP. Indeed, just above D_{lc}, one has $D - 2 + \eta_{cr} \ll 1$ so that the nontrivial solution of Eq.(7) obeys $\alpha_{cr} \ll 1$ and thus $\zeta_{cr} \gg 1$ which is precisely the regime of validity of this equation. Thus, just above D_{lc}, Eq.(7) well describes both the CTFP and the FLFP which coexist and D_{lc} corresponds to the dimension at which the CTFP collapses to the FLFP which becomes unstable. Using the relation $\eta(D_{lc}, u_f, v_f) = 2 - D_{lc}$ one obtains:

$$d = \frac{D_{lc}^4 + 6D_{lc}^3 - 3D_{lc}^2 + 4D_{lc}}{2(-D_{lc}^2 - D_{lc} + 6)} \qquad (8)$$

which, once inversed, provides the expression of $D_{lc}(d)$. The corresponding curve is displayed on the left part of Fig.1. In particular one has, for genuine membranes: $D_{lc}(d = 3) \simeq 1.33$. This result displays a remarkable stability with respect to a change in the cut-off function. With $R_k(q) = Zq^4/(\exp(q^4/k^4) - 1)$ one finds $D_{lc}(d = 3) \simeq 1.30$. Our results compare well with the large-d, $D_{lc}(d = 3) = 4/3$ [10], and SCSA, $D_{lc}(d = 3) = 1.5$ [16].

In summary, we have investigated the crumpling transition and flat phase of D-dimensional polymerized membranes embedded in a d-dimensional space within a nonperturbative RG approach. We have determined the whole line $d_{cr}(D)$ that separates the second- and first-order regions, the lower critical dimension $D_{lc}(d)$ and the critical exponents. More sophisticated ansatz should be used to systematically increase the accuracy of our results although implying a heavy algebra. Finally, our approach can be applied to many other situations in which the perturbative approaches lead to an unsatisfying quantitative or even qualitative description such as in self-avoiding, anisotropic or disordered membranes [1, 2, 26].

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